Beam parameterization and invariants in a periodic solenoidal channel*

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The standard Courant-Snyder formalism is generalized for a solenoidal channel, for which 4D instead of 2D treatment is required due to the cylindrical symmetry. Unlike a coupled quadrupole channel, beam dynamics in a solenoidal channel have unique properties. Four quadratic constants of the motion, the canonical angular momentum plus three Courant-Snyder type invariants, are found. Based on this, a natural set of beam-envelope functions are introduced to parameterize the ten independent beam moments of the 4D phase-space distribution of a beam. General expression of an equilibrium Gaussian distribution is given. General formalism is developed for describing the evolution of matched and mismatched beams in periodic solenoidal channels.

29.27.Bd, 41.85.-p, 29.27.-a, 41.75.-i

I. INTRODUCTION

Solenoids are among the basic magnetic devices that provide transverse focusing to a charged beam. Focusing channels made of solenoids can provide simultaneous focusing for both transverse planes. Thus they are commonly used for transporting low-energy beams [1]. Although usually not as effective as quadrupoles for small-aperture beams at higher energies, solenoids may still be the best choice for a beam of very large size. Recently beam dynamics in a solenoidal channel have been attracting research attention since most of the beam focusing systems in the envisioned muon collider [2] and neutrino factory [3] are solenoidal channels. Evolution of a cylindrical symmetric beam in a solenoidal ionization cooling channel has been solved in Ref. [4]. In this paper we consider the evolution of a *general* monochromatic beam injected on the axis of an ideal, cylindrically symmetric, solenoidal channel. Beam parameterization used in Ref. [4] is generalized here to cover nonsymmetric beams.

To describe the beam evolution in a focusing channel, second-order-beam-moment and beam-envelope equations are often used because (a) the second moments of beam phase-space distribution contain most of the important beam properties such as rms beam size and angular divergence; and (b) the evolution equations of second moments close on themselves for linear dynamics that often dominate beam evolution. There are ten independent second moments for the two transverse degrees of freedom. It is important to choose the right parameterization to characterize these moments so that it provides convenient description of beam properties and evolution. The focus of this paper is on the parameterization of beam moments. Symmetries and associated invariants play a key role in the choice of our parameters, based on which general formalism for beam evolution under the influence of linear focusing forces in solenoidal channels is presented.

The cylindrical symmetry results in unique properties for phase-space dynamics of a solenoidal channel. In addition to the well-known Courant-Snyder invariants for the two subspaces, there are two more linearly independent quadratic invariants: the obvious angular momentum and a more subtle Courant-Snyder type invariant for the cross space. Unlike in a quadrupole channel, instead of two there is only one independent betatron phase because the phase difference between the two independent subspaces is fixed by the angular momentum conservation. As a result, 4D phase-space beam parameterization is required.

This paper is organized as follows. Section II reviews the standard Courant-Synder theory for one degree of freedom to emphasize the fundamental concepts and to make the following sections easy to read. Section III discusses single-particle dynamics and constants of the motion in a solenoidal channel. Section IV discusses beam-moment invariants, equilibrium beam distribution, and its parameterization. Section V discusses parameterization of a general beam and the filamentation process via which an unmatched beam evolves into an equilibrium beam. Appendix A offers another viewpoint to appreciate the beam phase-space structure and our parameterization. Appendix B proves that the four invariants given in section III are the only linearly independent quadratic invariants.

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II. REVIEW OF COURANT-SYNDER THEORY FOR ONE DEGREE OF FREEDOM

In a one-degree-of-freedom focusing system, the transverse motion of a particle is governed by

$$x''(s) + K(s)x(s) = 0 \tag{1}$$

where the prime means derivative with respect to the longitudinal position s, and $K(s)$ describes the periodic focusing forces in the channel. The corresponding Hamiltonian is

$$H = \frac{1}{2}\left[\, P_x^2 + K(s)\, x^2 \,\right]. \tag{2}$$

The general solution in terms of action-angle variables (J, ψ) is

$$x = \sqrt{2J\hat{\beta}(s)}\cos\psi, \tag{3}$$

$$P_x = -\sqrt{\frac{2J}{\hat{\beta}(s)}}\left[\sin\psi + \hat{\alpha}(s)\cos\psi\right], \tag{4}$$

where the lattice functions $\hat{\beta}$ and $\hat{\alpha}$ are the solutions of

$$\hat{\beta}' = -2\,\hat{\alpha}, \quad \hat{\alpha}' = K(s)\hat{\beta} - \hat{\gamma}, \quad \hat{\gamma} = \frac{1+\hat{\alpha}^2}{\hat{\beta}} \tag{5}$$

with periodic boundary condition. The angle $\psi = \psi(0) + \Psi(s)$, the sum of the initial angle and the phase advance

$$\Psi(s) = \int_0^s \frac{1}{\hat{\beta}(\bar{s})}\, d\bar{s}. \tag{6}$$

The phase-space motion of a particle can be conveniently described by the transfer matrix $M(s)$ that maps the phase-space coordinate $X \equiv \{x, P_x\}^T$ from an initial $X(0)$ to $X(s) = M(s)X(0)$ at location s. Using lattice functions defined in Eqs. (5,6), the general form of the transfer matrix can be written as

$$M(s) = A(s)R(\Psi(s))A(0)^{-1}, \tag{7}$$

where the matrix

$$A(s) = \begin{bmatrix} \sqrt{\hat{\beta}(s)} & 0 \\ -\frac{\hat{\alpha}(s)}{\sqrt{\hat{\beta}(s)}} & \frac{1}{\sqrt{\hat{\beta}(s)}} \end{bmatrix} \tag{8}$$

is the well-known transformation from Floquet space to the normal phase space, and the matrix $R(\Psi(s))$ represents a rotation in the Floquet-space by an angle $\Psi(s)$ that is the betatron phase advance:

$$R(\Psi) = \begin{bmatrix} \cos\Psi & \sin\Psi \\ -\sin\Psi & \cos\Psi \end{bmatrix}. \tag{9}$$

The most important feature of single-particle motion is the existence of the well-known Courant-Snyder invariant $I = \hat{\gamma}x^2 + 2\hat{\alpha}xP_x + \hat{\beta}P_x^2$ [5]. This quadratic invariant defines a "machine ellipse" in the phase space for stable particles to move on. The Floquet transformation maps a machine ellipse onto a circle in Floquet space and the particle's motion becomes a simple rotation.

For a beam distribution, the standard parameterization for beam moments are

$$\langle x^2\rangle \equiv \epsilon\,\beta,\ \langle xP_x\rangle \equiv -\epsilon\,\alpha,\ \langle P_x^2\rangle \equiv \epsilon\,\gamma, \tag{10}$$

and the rms emittance $\epsilon \equiv \sqrt{\langle x^2\rangle\langle P_x^2\rangle - \langle xP_x\rangle^2}$ for the x-P_x canonical phase space. Here the angular brackets represent average over beam phase-space distribution. By definition, $1 + \alpha^2 = \beta\gamma$. Thus three independent parameters ϵ, β, and α are used to parameterize the three independent moments. The significance of these parameters can be appreciated by that fact the equi-density (1σ) contour of a general Gaussian distribution can be written as $\gamma x^2 + 2\alpha xP_x + \beta P_x^2 = \epsilon$. Thus β, α, and γ define a "beam ellipse" that reflects the shape of the distribution and emittance ϵ reflects the density of the distribution ($\pi\epsilon$ is equal to the area of the ellipse). Furthermore, the emittance is conserved for linear Hamiltonain dynamics.

The beam-envelope functions β, α, and γ satisfy the same differential equation, Eq. (5), as the lattice functions $\hat{\beta}$, $\hat{\alpha}$, and $\hat{\gamma}$. However, the boundary condition for the envelope functions are determined by the beam distribution at the entrance. Hence the beam envelope functions are in general distinct from the lattice functions. Nonetheless, because each particle must move on the machine ellipse, the beam ellipse will match onto the machine ellipse when the beam distribution reaches equilibrium via filamentation. Thus for a matched beam (a beam in equilibrium state) $\beta = \hat{\beta}$, $\alpha = \hat{\alpha}$, and $\gamma = \hat{\gamma}$. To understand this process better, let us assume, at the beginning, the phase-space distribution is given by the beam moment matrix

$$\Sigma(0) = \begin{bmatrix} \langle x^2\rangle & \langle xP_x\rangle \\ \langle xP_x\rangle & \langle P_x^2\rangle \end{bmatrix}_{s=0} = \epsilon \begin{bmatrix} \beta(0) & -\alpha(0) \\ -\alpha(0) & \gamma(0) \end{bmatrix}. \tag{11}$$

Transformed to the Floquet space, we have

$$\bar{\Sigma}(0) = A(0)^{-1}\Sigma(0)\left(A(0)^{-1}\right)^T = \epsilon \begin{bmatrix} \bar{\beta} & -\bar{\alpha} \\ -\bar{\alpha} & \bar{\gamma} \end{bmatrix}, \tag{12}$$

where

$$\bar{\beta} = \frac{\beta(0)}{\hat{\beta}(0)},\ \bar{\alpha} = \alpha(0) - \frac{\beta(0)}{\hat{\beta}(0)}\hat{\alpha}(0),\ \bar{\gamma} = \frac{1+\bar{\alpha}^2}{\bar{\beta}}. \tag{13}$$

Here we use the overbar to indicate quantities in the Floquet space. Note that, for a matched beam, $\bar{\beta} = \bar{\gamma} = 1$, $\bar{\alpha} = 0$, i.e., the beam ellipse is a circle in the Floquet space [6]. To propagate the beam ellipse to location s in the Floquet space, we simply need to rotate the beam as

$$\bar{\Sigma}(s) = R(\Psi(s))\bar{\Sigma}(0)R(\Psi(s))^T \tag{14}$$
$$= \epsilon\left\{\frac{\bar{\beta}+\bar{\gamma}}{2} - \bar{\alpha}\begin{bmatrix}\sin(2\Psi) & \cos(2\Psi)\\ \cos(2\Psi) & -\sin(2\Psi)\end{bmatrix} + \frac{\bar{\beta}-\bar{\gamma}}{2}\begin{bmatrix}\cos(2\Psi) & -\sin(2\Psi)\\ -\sin(2\Psi) & -\cos(2\Psi)\end{bmatrix}\right\}.$$

Transforming back to the normal phase space using $\Sigma(s) = A(s)\bar{\Sigma}(s)A(s)^T$, one can express the beam moments as functions of machine parameters $\hat{\beta}(s)$, $\hat{\alpha}(s)$, $\hat{\gamma}(s)$, $\Psi(s)$ and the initial beam distribution parameters ϵ, $\beta(0)$, $\alpha(0)$, and $\gamma(0)$. However we will not spell it out here.

Because of the last two terms in Eq. (14), the beam envelope of an unmatched beam will oscillate at two times the betatron oscillation frequency. Due to nonlinearity, different particles experience different phase advances, and soon the phase advances become evenly distributed, thus the last two terms in $\bar{\Sigma}(s)$ average to zero. This is the process of filamentation. For a fully filamented beam, $\bar{\Sigma}(s) \to \epsilon\frac{\bar{\beta}+\bar{\gamma}}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, thus the beam becomes matched but with an enlarged emittance. Figure 1 illustrates the enlargement of emittance after filamentation. The emittance magnification factor

$$B_{mag} \equiv \frac{\bar{\beta}+\bar{\gamma}}{2} = -\alpha(0)\hat{\alpha}(0) + \frac{\beta(0)\hat{\gamma}(0)+\gamma(0)\hat{\beta}(0)}{2}$$
$$= \frac{1}{2}\left\{\frac{\beta}{\hat{\beta}} + \frac{\hat{\beta}}{\beta} + \left(\sqrt{\frac{\beta}{\hat{\beta}}}\hat{\alpha} - \sqrt{\frac{\hat{\beta}}{\beta}}\alpha\right)^2\right\}_{s=s_0} \tag{15}$$

is the figure of merit to evaluate emittance degradation due to beam mismatch [7,6,8].

To summarize, in one-degree-of-freedom periodic focusing channels, after filamentation, a beam will reach an equilibrium distribution whose beam ellipse matches the machine ellipse determined by the Courant-Synder invariant $I = \hat{\gamma}x^2 + 2\hat{\alpha}xP_x + \hat{\beta}P_x^2$. In the following, we will generalize this formalism to the two-degree-of-freedom solenoidal channels.

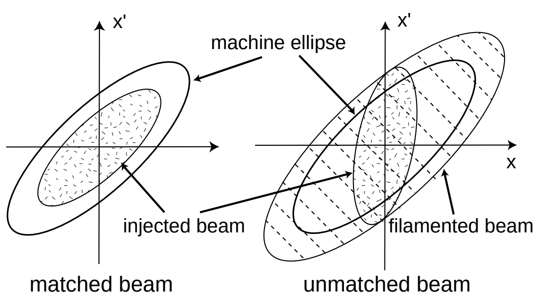

FIG. 1. Illustration of emittance degradation due to mismatch between injected beam ellipse and machine ellipse.

III. SINGLE-PARTICLE DYNAMICS AND CONSTANTS OF THE MOTION IN A SOLENOIDAL CHANNEL

Let s be the coordinate along the solenoid axis, x and y be the transverse coordinates. Due to the cylindrical symmetry, the magnetic field in a solenoid is determined by the on-axis field $B(s)$. Close to the axis, especially for a slowly varying $B(s)$, the field is linear and given by

$$\mathbf{B}(x,y,s) = B(s)\mathbf{e}_z - \frac{1}{2}B'(s)(x\mathbf{e}_x + y\mathbf{e}_y) + \cdots, \tag{16}$$

where the prime means derivative with respect to s and $\mathbf{e}_z$ is the unit vector in the s-direction, etc. It is not difficult to work out the equation of motion and Hamiltonian that govern the single particle dynamics. The Hamiltonian H for linear transverse dynamics of a single particle with charge q and longitudinal momentum P_s in the laboratory frame is [9]

$$H = \frac{1}{2}\left(P_x^2 + P_y^2\right) + \frac{1}{2}\kappa^2(s)\left(x^2+y^2\right) + \kappa(s)\,L_z, \tag{17}$$

where $\kappa(s) = qB(s)/2P_s$, P_x and P_y are the canonical momenta, and $L_z = xP_y - yP_x$ is the canonical angular momentum.

Notice that the Hamiltonian in the laboratory frame is coupled between x and y due to the angular momentum term. However it is well known that in the Larmor frame rotating about the z-axis at Larmor frequency (one half of the cyclotron frequency) [10,11], the Hamiltonian is decoupled and becomes [9]

$$\tilde{H} = \frac{1}{2}\left[\tilde{P}_x^2 + \kappa^2(s)\,\tilde{x}^2\right] + \frac{1}{2}\left[\tilde{P}_y^2 + \kappa^2(s)\,\tilde{y}^2\right]. \tag{18}$$

Hereafter, we will use the tilde over a symbol to indicate that it is in the rotating frame. To make the frame transformation explicit, let $R_{\hat{z}}(\theta)$ represent the rotation about the z-axis that transforms $\{x, P_x, y, P_y\}$ into $\{\tilde{x}, \tilde{P}_x, \tilde{y}, \tilde{P}_y\}$. Then

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = R_{\hat{z}}(\theta)\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}, \tag{19}$$

and in the same way P_x and P_y are rotated. This rotation can be generated by the generating function $F_2(x, \tilde{P}_x, y, \tilde{P}_y) = xP_x + yP_y$. For the Larmor frame, the rotation angle is given by

$$\theta_L(s) = -\int_0^s \kappa(\bar{s})\,d\bar{s}. \tag{20}$$

Note that the rotation angle could be shifted by an arbitrary constant.

From Eq. (18) we see that, in the Larmor frame, due to the cylindrical symmetry of solenoids, the two uncoupled degrees of freedom have exactly the same Hamiltonian. Furthermore the Hamiltonian has the same structure as in a quadrupole channel and can be treated with the standard Courant-Snyder formalism. Therefore we can *naturally* introduce *one* set of "machine ellipse" parameters $\{\hat{\beta}, \hat{\alpha}, \hat{\gamma}\}$ for a solenoidal channel via the well-known Courant-Snyder invariants for the two subspaces:

$$I_x = \hat{\gamma}(s)\tilde{x}^2 + 2\hat{\alpha}(s)\tilde{x}\tilde{P}_x + \hat{\beta}(s)\tilde{P}_x^2, \tag{21}$$

$$I_y = \hat{\gamma}(s)\tilde{y}^2 + 2\hat{\alpha}(s)\tilde{y}\tilde{P}_y + \hat{\beta}(s)\tilde{P}_y^2. \tag{22}$$

The machine parameters $\hat{\beta}$, $\hat{\alpha}$, and $\hat{\gamma}$ are determined by Eq. (5) with $K(s) = \kappa(s)^2$. The existence of the two independent invariants I_x and I_y is due to the fact that there are two decoupled subspaces, which is manifested by the Hamiltonian in the Larmor frame. Obviously the canonical angular momentum L_z is another constant of the motion due to the cylindrical symmetry. Recalling that the Poisson bracket of any two constants of the motion is a constant of the motion as well [12], it is easy to check that $\{L_z, I_x\} = -\{L_z, I_y\}$ yields one more constant of the motion

$$I_{xy} = \hat{\gamma}(s)\,\tilde{x}\,\tilde{y} + 2\hat{\alpha}(s)\frac{\tilde{x}\tilde{P}_y + \tilde{y}\tilde{P}_x}{2} + \hat{\beta}(s)\tilde{P}_x\tilde{P}_y. \tag{23}$$

We have therefore found four invariants I_x, I_y, I_{xy}, and L_z for the single-particle dynamics. Furthermore, all these invariants are quadratic forms of the dynamical variables. In Appendix B we show that these four invariants are the only linearly independent quadratic invariants.[1] The orbit of a particle is completely determined by the values of these invariants and the machine parameters $\{\hat{\beta}, \hat{\alpha}, \hat{\gamma}\}$. These invariants result from the underlying symmetry. The complete symmetry group and geometric interpretation of particle motion can be established in the same way as in the case of the two-dimensional isotropic harmonic oscillator discussed in Ref. [12].

From Eq. (18) and the standard Courant-Snyder theory, we can simply write down the general form of the transfer matrix as

$$M(s) = R_{\hat{z}}^{-1}(\theta_L(s))\tilde{M}(s)R_{\hat{z}}(\theta_L(0)), \tag{24}$$

where the transfer matrix in the Larmor frame $\tilde{M}(s)$ is block diagonalized with two identical submatrices M_{2D} for the two decoupled transverse phase spaces, i.e.,

$$\tilde{M} = \begin{bmatrix} M_{2D} & 0 \\ 0 & M_{2D} \end{bmatrix}, \tag{25}$$

where M_{2D} has the form of Eq. (7). See Appendix A for explicit expressions of a particle orbit and the four invariants in terms of action-angle variables.

IV. BEAM MOMENTS PARAMETERIZATION OF EQUILIBRIUM DISTRIBUTION

Now we turn our attention to the properties of a beam—the second moments of phase-space distribution. From the four invariants found above, it is straightforward to construct the matched beam moments (see Appendix A for more discussion). Analogous to the 2D case in Eq. (10), the ten matched beam moments can be parameterized as

$$\{\langle\tilde{x}^2\rangle, \langle\tilde{x}\tilde{P}_x\rangle, \langle\tilde{P}_x^2\rangle\} = \epsilon_x\{\hat{\beta}(s), -\hat{\alpha}(s), \hat{\gamma}(s)\}, \tag{26}$$

$$\{\langle\tilde{y}^2\rangle, \langle\tilde{y}\tilde{P}_y\rangle, \langle\tilde{P}_y^2\rangle\} = \epsilon_y\{\hat{\beta}(s), -\hat{\alpha}(s), \hat{\gamma}(s)\}, \tag{27}$$

$$\{\langle\tilde{x}\tilde{y}\rangle, \frac{\langle\tilde{x}\tilde{P}_y\rangle + \langle\tilde{y}\tilde{P}_x\rangle}{2}, \langle\tilde{P}_x\tilde{P}_y\rangle\} = \epsilon_{xy}\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}, \tag{28}$$

$$\langle\tilde{x}\tilde{P}_y\rangle - \langle\tilde{y}\tilde{P}_x\rangle = \langle L_z\rangle \equiv L, \tag{29}$$

$$\epsilon_x \equiv \sqrt{\langle\tilde{x}^2\rangle\langle\tilde{P}_x^2\rangle - \langle\tilde{x}\tilde{P}_x\rangle^2} = \frac{1}{2}\langle I_x\rangle, \tag{30}$$

$$\epsilon_y \equiv \sqrt{\langle\tilde{y}^2\rangle\langle\tilde{P}_y^2\rangle - \langle\tilde{y}\tilde{P}_y\rangle^2} = \frac{1}{2}\langle I_y\rangle, \tag{31}$$

$$\epsilon_{xy} \equiv \sqrt{\langle\tilde{x}\tilde{y}\rangle\langle\tilde{P}_x\tilde{P}_y\rangle - \langle\frac{\tilde{x}\tilde{P}_y + \tilde{y}\tilde{P}_x}{2}\rangle^2} = \frac{1}{2}\langle I_{xy}\rangle, \tag{32}$$

where $\langle I_x\rangle$, $\langle I_y\rangle$, $\langle I_{xy}\rangle$, and $\langle L_z\rangle$ are the phase-space averages of the single-particle invariants, and thus they obviously are beam invariants. In terms of the beam-moment matrix of phase-space distribution $\tilde{\Sigma} \equiv \langle\tilde{X}(s)\tilde{X}(s)^T\rangle$, where $\tilde{X}^T = \{\tilde{x}, \tilde{P}_x, \tilde{y}, \tilde{P}_y\}$ is the transpose of the phase-space vector $\tilde{X}$, the parameterization in Eqs. (26–29) can be written as

$$\tilde{\Sigma} \equiv \begin{bmatrix} \epsilon_x \mathrm{T} & \epsilon_{xy}\mathrm{T} + \frac{L}{2}\mathrm{J} \\ \epsilon_{xy}\mathrm{T} + \frac{L}{2}\mathrm{J}^T & \epsilon_y \mathrm{T} \end{bmatrix}, \tag{33}$$

where

$$\mathrm{T} \equiv \begin{bmatrix} \hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha} & \hat{\gamma} \end{bmatrix} \quad \text{and} \quad \mathrm{J} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{34}$$

For a Gaussian beam, the matched beam distribution is completely determined by the $\tilde{\Sigma}$-matrix via

$$\rho(\tilde{X}) = \frac{1}{(2\pi)^2\sqrt{\det(\Sigma)}}\, e^{-\frac{1}{2}\tilde{X}^T\tilde{\Sigma}^{-1}\tilde{X}} = \frac{1}{(2\pi)^2\sqrt{\epsilon_{4\mathrm{D}}}}\, e^{-\frac{\epsilon_y I_x + \epsilon_x I_y - 2\epsilon_{xy}I_{xy} - LL_z}{2(\epsilon_x\epsilon_y - \epsilon_{xy}^2 - L^2/4)}}, \tag{35}$$

[1] Since in a system of two degrees of freedom there can at most be only three such algebraic constants of the motion [12], these four invariants must be related. It is not difficult to show that they are related nonlinearly as:

$$L_z^2 + I_{xy}^2 + \left(\frac{I_x - I_y}{2}\right)^2 = \left(\frac{I_x + I_y}{2}\right)^2.$$

and the 4D emittance is

$$\epsilon_{4\mathrm{D}} \equiv \det \Sigma = \det \tilde{\Sigma} = (\epsilon_x \epsilon_y - \epsilon_{xy}^2 - L^2/4)^2. \quad (36)$$

Note that the distribution in Eq. (35) depends on the phase-space coordinates via the four invariants only, and thus $\rho(\tilde{X})$ is an equilibrium distribution. In fact, it is the most general form of equilibrium Gaussian distribution in a solenoidal channel because the invariant quadratic form in any equilibrium Gaussian distribution must be a linear combination of the four invariants (Appendix B).

The parameterization given in Eq. (33) is based on and emphasizes the two ($\tilde{x}$ and $\tilde{y}$) obviously independent subspaces. Another slightly different parameterization can be used to emphasize the cylindrical symmetry by introducing the cylindrically symmetric and asymmetric emittaces ϵ_s and ϵ_a via

$$\{\langle\tilde{x}^2\rangle+\langle\tilde{y}^2\rangle, \langle\tilde{x}\tilde{P}_x\rangle+\langle\tilde{y}\tilde{P}_y\rangle, \langle\tilde{P}_x^2\rangle+\langle\tilde{P}_y^2\rangle\} = 2\epsilon_s\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}, \quad (37)$$

$$\{\langle\tilde{x}^2\rangle-\langle\tilde{y}^2\rangle, \langle\tilde{x}\tilde{P}_x\rangle-\langle\tilde{y}\tilde{P}_y\rangle, \langle\tilde{P}_x^2\rangle-\langle\tilde{P}_y^2\rangle\} = 2\epsilon_a\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}. \quad (38)$$

Note that L and ϵ_{xy} already correspond to the symmetric and asymmetric parts respectively in the cross space.

Rotating back to the laboratory frame, we obtain the evolution of moments for a matched beam as

$$\{\langle x^2\rangle, \langle xP_x\rangle, \langle P_x^2\rangle\} = (\epsilon_s + \sqrt{\epsilon_a^2+\epsilon_{xy}^2}\sin\Theta)\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}, \quad (39)$$

$$\{\langle y^2\rangle, \langle yP_y\rangle, \langle P_y^2\rangle\} = (\epsilon_s - \sqrt{\epsilon_a^2+\epsilon_{xy}^2}\sin\Theta)\{\hat{\beta}, -\hat{\alpha}, \hat{\gamma}\}, \quad (40)$$

$$\{\langle xy\rangle, \langle P_xP_y\rangle\} = \sqrt{\epsilon_a^2+\epsilon_{xy}^2}\cos\Theta\,\{\hat{\beta}(s), \hat{\gamma}(s)\}, \quad (41)$$

$$\langle xP_y\rangle = \quad L/2 - \sqrt{\epsilon_a^2+\epsilon_{xy}^2}\cos\Theta\,\hat{\alpha}(s), \quad (42)$$

$$\langle yP_x\rangle = -L/2 - \sqrt{\epsilon_a^2+\epsilon_{xy}^2}\cos\Theta\,\hat{\alpha}(s). \quad (43)$$

Here the rotating angle

$$\Theta(s) = 2\theta_L - \arctan(\epsilon_a/\epsilon_{xy}). \quad (44)$$

The machine-ellipse parameters $\hat{\beta}$, $\hat{\alpha}$, and $\hat{\gamma}$ are determined by the solenoidal field via Eq. (5), while the emittances ϵ_s, ϵ_a, ϵ_{xy} and the angular momentum L are given by the incoming beam distribution. Among these ten moments, $\langle x^2\rangle$, $\langle y^2\rangle$, and $\langle xy\rangle$ are the most useful ones since they are readily available from beam profile measurements. Note that these three spatial moments provide information to determine the three emittances ϵ_s, ϵ_a, and ϵ_{xy}, but not the angular momentum [13]. Explicitly,

$$\epsilon_s = \left(\langle x^2\rangle + \langle y^2\rangle\right)/2\hat{\beta}(s), \quad (45)$$

$$\epsilon_a^2 + \epsilon_{xy}^2 = \left[\left(\langle x^2\rangle - \langle y^2\rangle\right)^2 + 4\langle xy\rangle^2\right] \Big/ 4\hat{\beta}^2(s), \quad (46)$$

$$\frac{\epsilon_a}{\epsilon_{xy}} = \tan\left[2\theta_L - \cot^{-1}\frac{2\langle xy\rangle}{\langle x^2\rangle - \langle y^2\rangle}\right]. \quad (47)$$

Furthermore, the maximum and minimum rms beam sizes are $\sqrt{\left(\epsilon_s \pm \sqrt{\epsilon_a^2+\epsilon_{xy}^2}\right)\hat{\beta}(s)}$.

An important special equilibrium distribution is the cylindrical-symmetric beam for which $\epsilon_s = \epsilon_x = \epsilon_y$ and $\epsilon_a = \epsilon_{xy} = 0$. The beam-moment parameterizations in both the laboratory and the Larmor frames are then reduced to

$$\Sigma_s = \tilde{\Sigma}_s = \begin{bmatrix} \epsilon_s \mathrm{T} & \frac{L}{2}\mathrm{J} \\ \frac{L}{2}\mathrm{J}^T & \epsilon_s \mathrm{T} \end{bmatrix}. \quad (48)$$

This parameterization played a key role in solving the dynamics of ionization cooling in a solenoidal channel [4]. As shown above, Eq. (48) is a special case of the general parameterization based on the symmetries and constants of the motion in solenoidal channels. A somewhat different parameterization for a cylindrical symmetric beam was employed in Refs. [14,15].

V. GENERAL PARAMETERIZATION AND EVOLUTION OF A MISMATCHED BEAM

Based on the standard Courant-Snyder theory for a quadrupole channel, and using the four constants of the motion, we have discussed the general single-particle dynamics and evolution of a matched beam in a solenoidal channel. Note that there are four independent parameters $\{\epsilon_s,\ L,\ \hat{\beta},$ and $\hat{\alpha}\}$ for a matched cylindrically-symmetric beam and, two additional parameters $\{\epsilon_a, \epsilon_{xy}\}$ for a general matched beam. For an arbitrary Gaussian beam, there are ten independent second moments. To generalize our parameterization to cover the unmatched beams, four more parameters are needed. In general, the beam-envelope functions β and α for unmatched beams are not the same as the machine parameters, thus we need different sets of beam-envelope functions for different subspaces associated with the three emittances. Therefore we generalize the parameterization in Eqs. (26,27,28) as follows:

$$\{\langle\tilde{x}^2\rangle, \langle\tilde{x}\tilde{P}_x\rangle, \langle\tilde{P}_x^2\rangle\} = \epsilon_x\{\beta_x(s), -\alpha_x(s), \gamma_x(s)\}, \quad (49)$$

$$\{\langle\tilde{y}^2\rangle, \langle\tilde{y}\tilde{P}_y\rangle, \langle\tilde{P}_y^2\rangle\} = \epsilon_y\{\beta_y(s), -\alpha_y(s), \gamma_y(s)\}, \quad (50)$$

$$\{\langle\tilde{x}\tilde{y}\rangle, \frac{\langle\tilde{x}\tilde{P}_y\rangle + \langle\tilde{y}\tilde{P}_x\rangle}{2}, \langle\tilde{P}_x\tilde{P}_y\rangle\} = \epsilon_{xy}\{\beta_{xy}, -\alpha_{xy}, \gamma_{xy}\}, \quad (51)$$

where six different envelope functions are introduced to replace the two machine functions used for the matched beams. Using this parameterization, the general form of the beam-moment matrix $\tilde{\Sigma}$ can be written as

$$\tilde{\Sigma} = \begin{bmatrix} \tilde{\Sigma}_x & \tilde{\Sigma}_{xy} + \frac{L}{2}J \\ \tilde{\Sigma}_{xy} + \frac{L}{2}J^T & \tilde{\Sigma}_y \end{bmatrix}, \quad (52)$$

where $\tilde{\Sigma}_x = \epsilon_x \begin{bmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{bmatrix}$, $\tilde{\Sigma}_y = \epsilon_y \begin{bmatrix} \beta_y & -\alpha_y \\ -\alpha_y & \gamma_y \end{bmatrix}$, and $\tilde{\Sigma}_{xy} = \epsilon_{xy} \begin{bmatrix} \beta_{xy} & -\alpha_{xy} \\ -\alpha_{xy} & \gamma_{xy} \end{bmatrix}$. The parameterization in Eqs. (37,38) can be generalized similarly. It is not difficult to show that all these envelope functions satisfy Eq. (5).

Now let us study the evolution of the ten beam moments of an unmatched beam. Using the linear transfer matrix $\tilde{M}(s)$, the evolution of beam moments in the Larmor frame can be written as

$$\tilde{\Sigma}(s) = \langle \tilde{X}(s)\tilde{X}(s)^T \rangle = \tilde{M}(s)\tilde{\Sigma}(0)\tilde{M}(s)^T. \tag{53}$$

Due to the special form of the transfer matrix in Eq. (25), each 2D sub-block of the beam-moment matrix $\tilde{\Sigma}$ evolves independently and is governed by the same transfer matrix M_{2D}, i.e.,

$$\tilde{\Sigma}_x(s) = M_{2\mathrm{D}}(s)\,\tilde{\Sigma}_x(0)\,M_{2\mathrm{D}}(s)^T, \tag{54}$$

$$\tilde{\Sigma}_y(s) = M_{2\mathrm{D}}(s)\,\tilde{\Sigma}_y(0)\,M_{2\mathrm{D}}(s)^T, \tag{55}$$

$$\tilde{\Sigma}_{xy}(s) + \frac{L}{2}J = M_{2\mathrm{D}}(s)\,\tilde{\Sigma}_{xy}(0)\,M_{2\mathrm{D}}(s)^T + \frac{L}{2}M_{2\mathrm{D}}(s)\,J\,M_{2\mathrm{D}}(s)^T. \tag{56}$$

Since the transfer matrix $M_{2\mathrm{D}}(s)$ must be a symplectic matrix, $M_{2\mathrm{D}}(s)\,J\,M_{2\mathrm{D}}(s)^T = J$ and thus the angular momentum term in the last equation can be dropped. We see that $\tilde{\Sigma}_x(s)$, $\tilde{\Sigma}_y(s)$, and $\tilde{\Sigma}_{xy}(s)$ evolve in exactly the same way. As described in section II, these beam moments will evolve and match into an equilibrium distribution of the solenoidal channel. Note that the emittance for each subspace has its own magnification factor that can be computed from Eq.(15) by replacing β and α with β_x and α_x, etc. The angular momentum is conserved and thus its magnification factor is unity. The filamentation speeds in all subspaces are the same since they all depend on the spread of the same phase advances. However, the emittance magnification factors are different in general and depend on the initial phase-space distribution.

VI. CONCLUSION

In summary, a general formalism is developed for the evolution of a single particle, a matched beam, and a mismatched beam in solenoidal channels. A general expression for the equilibrium Gaussian distribution in a periodic solenoidal channel is derived. Unique dynamical properties due to cylindrical symmetry are addressed. The proposed beam-envelope functions provide concise descriptions of beam behavior inside a solenoidal channel and the conditions to match a beam into and out of a channel.

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[1] See, for example, G.J. Caporaso and A.G. Cole, in *The Physics of Particle Accelerators* edited by M. Month and M. Dienes, AIP Conf. Proc. 249, vol. 2, 1992, p. 1615-1712.

[2] C. Ankenbradt et al., Phys. Rev. ST Accel. Beams **2**, 081001 (1999).

[3] "A Feasibility Study of a Neutrino Source Based on a Muon Storage Ring," edited by N. Holtkamp and D. Finley, March, 2000.

[4] K.-J. Kim and C.X. Wang, Phys. Rev. Lett. **85**, 760 (2000).

[5] E.D. Courant and H.S. Snyder, Ann. of Phys. **3**, 1 (1958).

[6] M. Sands, SLAC-AP-85.

[7] *Handbook of Accelerator Physics and Engineering*, p. 253, edited by A. Chao and M. Tigner, World Scientific Publishing Co. (1999).

[8] N. Merminga, et al., SLAC-PUB-5484.

[9] A.J. Dragt, Nucl. Instrum. and Methods in Phys. Res., Sect. A **298** (1990).

[10] R. P. Feynman, *The Feynman Lectures on Physics*, vol. II, p. 34-6, Addison-Wesley Publishing Company, 1964.

[11] See, for examples, J.D. Lawson, *The Physics of Charged-Particle Beams*, 2nd ed., Clarendon Press, Oxford, 1988; M. Reiser, *Theory and Design of Charged Particle Beams*, John Wiley & Sons, 1994.

[12] H. Goldstein, *Classical Mechanics*, 2nd ed., Addison-Wesley Publishing Company, 1980. (p. 407 and p. 423)

[13] It may be surprising that the beam angular momentum cannot be measured from beam profile measurement. To understand this point, we write, using the single-particle orbit given in Appendix A, $\langle \tilde{x}^2 \rangle = \langle J_x \rangle \hat{\beta} = \epsilon_x \hat{\beta}$, $\langle \tilde{y}^2 \rangle = \langle J_y \rangle \hat{\beta} = \epsilon_y \hat{\beta}$, and $\langle \tilde{x}\tilde{y} \rangle = \langle \sqrt{J_x J_y} \cos\Delta\psi \rangle \hat{\beta} = \epsilon_{xy}\hat{\beta}$. Note that even though $\langle \tilde{x}\tilde{y} \rangle$ may yield $\langle \cos\Delta\psi \rangle$, it is not sufficient to determine $\langle \sin\Delta\psi \rangle$ that is needed for the angular momentum $L = 2\langle \sqrt{J_x J_y} \sin\Delta\psi \rangle$.

[14] G. Penn and J.S. Wurtele, Phys. Rev. Lett. **85**, 764 (2000).

[15] G. Penn and J.S. Wurtele, Proceedings of LINAC2000.

APPENDIX A

To illustrate the beam phase-space structure introduced in Eq. (33), let us construct the beam-moment matrix starting from the single-particle motion. Using action-angle variables for each subspace, a particle's motion can be described as

$$\tilde{x} = \sqrt{2J_x\hat{\beta}}\cos\psi_x, \tag{A1}$$

$$\tilde{P}_x = -\sqrt{\frac{2J_x}{\hat{\beta}}}\left(\sin\psi_x + \hat{\alpha}\cos\psi_x\right), \tag{A2}$$

$$\tilde{y} = \sqrt{2J_y\hat{\beta}}\cos\psi_y, \tag{A3}$$

$$\tilde{P}_y = -\sqrt{\frac{2J_y}{\hat{\beta}}}\left[\sin\psi_y + \hat{\alpha}\cos\psi_y\right]. \tag{A4}$$

The single-particle invariants found in section III are

$$L_z = \sqrt{4J_xJ_y}\sin\Delta\psi, \tag{A5}$$

$$I_x = 2J_x,\ I_y = 2J_y,\ I_{xy} = \sqrt{4J_xJ_y}\cos\Delta\psi. \tag{A6}$$

Here we introduce the phase difference $\Delta\psi = \psi_x - \psi_y$ to respect the fact that it is a constant due to conservation of angular momentum. From this it is straightforward to write down this particle's contribution to the beam moments. Assuming the phase ψ_x is randomly distributed (as the result of filamentation, for example), we can carry out the average over ψ_x and get

$$\begin{bmatrix} J_x\mathrm{T} & J_1\mathrm{T} - J_2\mathrm{J} \\ J_1\mathrm{T} - J_2\mathrm{J}^T & J_y\mathrm{T} \end{bmatrix}, \tag{A7}$$

where $J_1 = \sqrt{J_xJ_y}\cos(\Delta\psi)$, $J_2 = \sqrt{J_xJ_y}\sin(\Delta\psi)$, and the matrices T and J are defined in Eq. (34). Averaging over the actions J_x and J_y yields the standard transverse emittances. If the phase difference $\Delta\psi$ is also randomly distributed, as in usual quadrupole channels, then J_1 and J_2 average to zero. However, in solenoidal channels, J_1 and J_2 will generally not average to zero but yield the "emittance" ϵ_{xy} for the "cross space" and the angular momentum term $-L/2$ in Eq. (33). Note that because of cylindrical symmetry in solenoidal channels, the angular momentum L_z is still conserved even with nonlinearity. Therefore $\Delta\psi$ of an individual particle as well as the distribution of $\Delta\psi$ in a beam will be conserved (instead of filamented).

APPENDIX B

To prove that the four quadratic invariants found in section III are the only linearly independent quadratic invariants in a solenoidal channel, we start with the Hamiltonian $H = \frac{1}{2}[P_x^2 + k(s)x^2] + \frac{1}{2}[P_y^2 + k(s)y^2]$ and the most general quadratic form $I = X^TAX$, where X is the phase-space variables and A is a 4×4 symmetric matrix containing the coefficients. If I is an invariant,

$$\frac{dI}{ds} = \frac{\partial I}{\partial s} + \{I, H\} = 0, \tag{B1}$$

thus

$$X^T\frac{\partial A}{\partial s}X = \{H, X\}^TAX + X^TA\{H, X\}. \tag{B2}$$

The Poisson bracket $\{H, X\}$ can be easily evaluated with the Hamiltonian and becomes

$$\{H, X\} = [-P_x, k\,x, -P_y, k\,y] = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} X, \tag{B3}$$

where the matrix $B = \begin{bmatrix} 0 & -1 \\ k & 0 \end{bmatrix}$. Inserting this into Eq. (B2) and partitioning the matrix A into 2×2 submatrices, we get

$$\frac{\partial A_{11}}{\partial s} = B^TA_{11} + A_{11}B \tag{B4}$$

for the submatrix A_{11} and the same form of equation for the submatrices A_{12} and A_{22}. Since these equations are decoupled, their contributions to the invariant are independent. Since A is symmetric, A_{11} and A_{22} must be symmetric. However $A_{12} = A_{21}^T$ can have any form. We can divide A_{12} into symmetric and antisymmetric parts. They are linearly independent and satisfy the same equation. It is straightforward to verify that the symmetric solution leads to the Courant-Snyder type of invariant. It is easy to see that the antisymmetric solution of A_{12} must be proportional to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which leads to the angular momentum. Therefore, there are exactly four linearly-independent quadratic invariants. Note that, if the focusing strengths in the two transverse planes are different, as in a quadrupole channel, there is no nonzero solution for the A_{12}, thus only two quadratic invariants exist.